**Let’s Circle It Up!**

Here are some example Circle problems that you can incorporate into your own classrooms!

1. **Points** $A$ **and** $B$ **lie on a circle centered at** $O$**, and** $∠AOB=60°$**. A second circle is internally tangent to the first and tangent to both** $\overbar{OA}$ **and** $\overbar{OB}$**. What is the ratio of the area of the smaller circle to that of the larger circle?**

***Solution:***

First let’s draw circle $O$.

$$B$$

$$C$$

$$P$$

$$Q$$

$$O$$

$$A$$

Now we need to draw circle $P$ circumscribed in circle $O$.

We need to create a right triangle. By doing so, we must bisect $∠AOB$. This creates line segment $\overbar{OC}$. We can denote the altitude, which is also the radius of circle $P$, as $\overbar{PQ}$.

Now we have created a $30°$-$60°$-$90°$ triangle, which is $∆POQ$. Let’s take a closer look at this triangle.



$$2r$$

$$r$$

$$r\sqrt{3}$$

$$Q$$

$$O$$

$$P$$

By using a theorem for the properties of $30°$-$60°$-$90°$ triangles, we know that the sides are in the ratio $1:\sqrt{3}:2$.

If we let $\overbar{OQ}$ be $r$, then by following this same theorem, we know that $\overbar{PO}=2r$ and $\overbar{PQ}=r\sqrt{3}$.

Now we need to find the radius of circle $O$. We will denote this by $r\_{o}$:

$$r\_{o}=\overbar{OC}=\overbar{OP}+\overbar{PC}$$

We also need to find the radius of circle $P$, which we already started to explain in the first diagram. This is denoted by $r\_{p}$:

$$r\_{p}=\overbar{PC}=\overbar{PQ}=r$$

Now, let’s solve for $r\_{o}$. We just stated that $r\_{o}=\overbar{OC}=\overbar{OP}+\overbar{PC}$, so now let’s substitute what we got using the theorem for $\overbar{OP}$ and $\overbar{PC}$.

$$r\_{o}=2r+r=3r$$

Now that we know the radius of circle $O$, we need to compute the areas of both circle $O$ and circle $P$. The area of circle $O$ will be denoted by $A\_{o}$ and the area of circle $P$ will be denoted by $A\_{p}$. To find the area, we will use the area formula, $A=πr^{2}$.

Starting with circle $O$, we just need to plug in what we got for its radius in for $r$.

$$A\_{o}=π\left(3r\right)^{2}=9r^{2}π$$

Following with circle $P$,

$$A\_{p}=π\left(r\right)^{2}=r^{2}π$$

If we set up a ratio of circle $P$ to circle $O$, we get

$$r^{2}π:9r^{2}π$$

We can see that both sides of the ratio consist of $r^{2}π$, so those terms cancel out, leaving us with a $1:9$ ratio of the two circle areas.

1. **Many Gothic cathedrals have windows with portions containing a ring of congruent circles that are circumscribed by a larger circle. In the figure shown, the number of smaller circles is four. What is the ratio of the sum of the areas of the four smaller circles to the area of the larger circle?**

***Solution:***

Let’s start by drawing the figure.

We need to let the four circumscribed circles be unit circles, meaning their radii will be one unit each. We know that the formula for an area of a circle is $A=πr^{2}$. Using this formula, we can compute the area of each small circle.

$$A\_{s}=π\left(1\right)^{2}$$

$$A\_{s}=π units^{2}$$

Since we have four small circles, we can multiply the area of one small circle by four. This gives us $4π units^{2}$.

Now we need to find the area of the larger circle. By doing so, we need to connect all the centers of the circles. This will form a square, because of symmetry.

We can use the Pythagorean Theorem to compute the diagonal in the square. We know each side of the square because it is just two radii added together, which in this case is 2. When we use the Pythagorean Theorem we get,

$$2^{2}+2^{2}=c^{2}$$

$$4+4=c^{2}$$

$$8=c^{2}$$

$$\pm \sqrt{8}=c$$

Therefore, the radius of the larger circle is $\sqrt{8}=2\sqrt{2}$.

We need to add 2 units to get the diameter of the larger circle.

$$d=2\sqrt{2}+2=2\left(\sqrt{2}+1\right) units^{2}$$

Therefore, the radius of the larger circle is $\sqrt{2}+1$.

Using this radius, we can plug this in the area of a circle formula.

$$A\_{L}=π\left(\sqrt{2}+1\right)^{2}$$

$$A\_{L}=π\left(3+2\sqrt{2}\right)$$

When set each area up as a ratio, we get

$$4π:\left(π\left(3+2\sqrt{2}\right)\right)$$

Since $π$ is on both sides, the terms cancel out, leaving us with

$$4:(3+2\sqrt{2})$$

1. **Let** $AB$ **be a linear segment and let** $P$ **be an arbitrary point on** $AB$**. Construct a semicircle** $S\_{1}$**with diameter** $AB$ **and a segment from** $P$ **that is perpendicular to** $AB$ **with an endpoint** $C$ **on** $S\_{1}$**. Prove that** $PC$ **is the geometric mean of** $AP$ **and** $BP$**.**

***Solution:***

First let’s draw the figure.

$$C$$

$$S\_{1}$$

$$l$$

$$m$$

$$n$$

$$b$$

$$a$$

$$B$$

$$A$$

$$P$$

The geometric mean of two numbers is found simply by taking the square root of their product.

We need to prove that $PC$ is the geometric mean of $AP$ and $BP$. Let’s say the length of $AP$ is $a$ and the length of $BP$ is $b$.

By connecting points $A$ and $C$ and points $B$ and $C$, we can use *Thales’ Theorem* to prove that $∆ACB$ is a right angle. Let the length of $AC=n$ and the length of $BC=m$.

*Thales’ Theorem*: The [diameter](http://www.mathopenref.com/diameter.html) of a circle always [subtends](http://www.mathopenref.com/subtend.html) a [right angle](http://www.mathopenref.com/angleright.html) to any point on the circle. If a triangle has, as one side, the [diameter](http://www.mathopenref.com/diameter.html) of a circle, and the third [vertex](http://www.mathopenref.com/vertex.html) of the triangle is any point on the [circumference](http://www.mathopenref.com/circumference.html) of the circle, then the triangle will always be a right triangle.

$∆ABC$ is represented as $n^{2}+m^{2}=\left(a+b\right)^{2}$, by the Pythagorean Theorem. Also, $∆APC$ and $∆BPC$ are right triangles as well because segment $PC$ is perpendicular to segment $AB$. Following the same concept, if we let the length of $PC=l$, then by using the Pythagorean theorem in $∆APC$ we get $a^{2}+l^{2}=n^{2}$ and in $∆BPC$ we get $b^{2}+l^{2}=m^{2}$. When we substitute our values for $n^{2}$ and $m^{2}$ from the above equations, we get $\left(a^{2}+l^{2}\right)+\left(b^{2}+l^{2}\right)=\left(a+b\right)^{2}$.

Now, let’s solve for $l$ which will give us the length of segment $PC$.

$$\left(a^{2}+l^{2}\right)+\left(b^{2}+l^{2}\right)=\left(a+b\right)^{2}$$

$$a^{2}+l^{2}+b^{2}+l^{2}=a^{2}+2ab+b^{2}$$

$$a^{2}+2l^{2}+b^{2}=a^{2}+2ab+b^{2}$$

$$2l^{2}=2ab$$

$$l^{2}=ab$$

$$l=\pm \sqrt{ab}$$

Earlier we stated that the geometric mean of two numbers was given by taking the square root of their product. We denoted this as $l=\sqrt{ab}$. Above, we have clearly stated and shown that $PC$ is the geometric mean of $AP$ and $BP$.

1. **A square and a circle intersect so that each side of the square contains a chord of the circle equal in length to the radius of the circle. What is the ratio of the area of the square to the area of the circle? Express your answer as a common fraction in terms of** $π$**.**

***Solution:***

We know that the area of a square is $A\_{□}=side^{2}$. Also, the area of a circle is $A\_{○}=πr^{2}$.

We can construct an equilateral triangle with the circle, using the radius. Note that the area of a triangle is $A\_{∆}=$ $\frac{1}{2}bh$.



We can use the Pythagorean Theorem to find the height of the triangle.

$$leg^{2}+leg^{2}=hypotenuse^{2}$$

$$\left(\frac{r}{2}\right)^{2}+h^{2}=r^{2}$$

$$\frac{r^{2}}{4}+h^{2}=r^{2}$$

$$h^{2}=\left(\frac{3}{4}\right)r^{2}$$

$$h=\pm \left(\frac{\sqrt{3}}{2}\right)r$$

$$2h=\sqrt{3}r=the side of the square$$

Using the Area of a Square formula, we get

$$A\_{□}=\left(\sqrt{3}r\right)^{2}$$

$$=3r^{2} units^{2}$$

Now, when we take the area of the square and divide it by the area of the circle we get,

$$\frac{A\_{□}}{A\_{○}}=\frac{3r^{2}}{πr^{2}}=\frac{3}{π}$$

Therefore, the ratio of the area of the square to the area of the circle is $\frac{3}{π}$.

1. **Given the shape below, which has the larger area white or gray?**

***Solution:***

Finding the area of the triangle: Let $x=the length of the base and height of the triangle.$

$$A\_{∆}=\frac{1}{2}\left(x\right)(x)$$

$$=\frac{x^{2}}{2} units^{2}$$

Finding the area of the black circle:

$$A\_{○}=π\left(\frac{x\sqrt{2}}{2}\right)^{2}$$

$$=π\left(\frac{x^{2}}{2}\right)$$

$$\frac{1}{2}A\_{○}= π\left(\frac{x^{2}}{4}\right)$$

Now, let’s subtract the area of the triangle from the area of half the circle to get the small black areas adjacent to the white triangle. We will denote this area as $A\_{a}$.

$$A\_{a}=\left(x^{2}∙\frac{π}{4}\right)-\left(\frac{x^{2}}{2}\right) units^{2}$$

We can now find the total area of the two gray *circles*.

$$A\_{○○}=\left(\frac{π}{8}x^{2}+\frac{π}{8}x^{2}\right)$$

By subtracting the area adjacent to the white triangle from the total area of the gray circles we are left with the shown gray crescent.

$$A\_{⌒}=\left(\frac{π}{8}x^{2}+\frac{π}{8}x^{2}\right)-\left(\frac{π}{4}x^{2}-\frac{x^{2}}{2}\right)$$

$$=\frac{2π}{8}x^{2}-\frac{π}{4}x^{2}+\frac{1}{2}x^{2}$$

$$=\frac{1}{2}x^{2} units^{2}$$

Therefore, we can see that the area of the white region is equal to the area of the gray region.

1. **Given the shape below, which color figure has the larger area? White or Black?**

***Solution:***

First, let’s find the area of the black circle.

$$R=r\sqrt{2}$$

$$r$$

$$r$$

$$A\_{PC}=πR^{2}$$

$$=π\left(r\sqrt{2}\right)^{2}$$

$$=2πr^{2} units^{2}$$

Now, let’s find the area of the white circle.

$$A\_{YC}=πr^{2}$$

In order to find the black area, we must subtract the white circle from the black circle.

$$A\_{PC}-A\_{YC}=\left(2πr^{2}\right)-(πr^{2})$$

$$=πr^{2}$$

Therefore, the area of the purple ring is equal to the area of the yellow circle.

1. **Two identical circles touch that same point of a rectangle- one from the inside, one from the outside. Both circles begin rolling in the plane along the perimeter of the rectangle until they return to the starting point. If the height of the rectangle is twice the circumference of the circles, and if the width is twice the height, how many revolutions will each circle make?**

***Solution:***

We know that one revolution is equal to the distance of the circumference.

We are given that the height ($h)$ is equal to two circumferences which is $2πd$. Also, the width ($w$) is twice the height. Thus, the width is $4πd$.

Next, we need to find the perimeter.

$$h+h+w+w=2πd+2πd+4πd+4πd=12πd units^{2}$$

So now that we know the perimeter, we need to take into consideration the outside and inside corners of the rectangle. How much does the circle move around each corner? In addition to the perimeter ($12πd$), the outside circle will also make a quarter rotation at each corner. Therefore, the outside circle will travel:

$$12πd+\frac{1}{4}πd+\frac{1}{4}πd+\frac{1}{4}πd+\frac{1}{4}πd=13πd$$

$$∴13 revolutions$$

Once the inside circle reaches a corner, it will have already covered one diameter of the given side. Therefore, the inside circle will travel:

$$12πd-4\left(1d\right)$$

$$=12πd-4\left(2r\right)$$

$$=12πd-8r$$

$$=12πd-8\left(\frac{πd}{2π}\right)$$

$$=πd\left(12-\frac{4}{π}\right)$$

$$∴\left(12-\frac{4}{π}\right)≈10.7 revolutions$$

So, the circles will make 10.7 revolutions around the rectangle.

1. **Given a penny and a circle that has twice the radius as the penny, after one revolution, how far has the penny traveled around the circle. How many revolutions does it take the penny to go fully around the circle?**

***Solution:***



We know that the radius of the penny is half the radius of the circle. Using a ruler we found that a US Penny is 0.75 inches in diameter. Therefore, the diameter of the circle is 1.5 inches. The circumference of the circle will be represented as $C=2πR$ and the circumference of the penny will be represented as $c=2πr$. Since the circle and penny are *similar*, we can set up a proportion.

$$\frac{c}{C}=\frac{r}{R}$$

After one revolution of the penny $(c)$, the penny will have traveled half the circumference of the circle. Thus, the penny will have traveled 2 revolutions to return to its starting point.