**Man Vs. Machine:**

**A Prime Example of Number Sense**

Adam Sprague

asprague@fredonia.edu

Stephanie Wisniewski

swisniewski@fredonia.edu



Man Vs. Machine is a group of activities that can be used in the classroom by teachers for 6th through 12th grade with the intention of using prime numbers to develop number sense. Throughout the grade levels, one of the most important jobs we have as math teachers is to develop students’ number sense. The importance of prime numbers, particularly prime factorization, tends to be overlooked in many classrooms. These activities are aimed at showing how the development of number sense can be derived from an understanding of prime numbers and prime factorization. These activities also emphasize the fact that a calculator is an important tool in the classroom, however is not a substitute for problem solving skills.

**Round 1:** Does the decimal representation of given unit fractions repeat or terminate?

Procedure:

* Start the activity by reviewing some terminology such as what it means for a decimal to repeat or terminate, what a unit fraction is, and a prime number is.
* Next, have a student get out his/her calculator and have another student select a unit fraction.
* Race against the student with the calculator to decide whether the decimal representation of this fraction will repeat or terminate.
* After winning a few rounds to the amazement of your students finally reveal to the students your number sense trick.

How it works:

 In order to beat the machine we need number sense and an understanding of prime factorization. A good place to start showing the importance of number sense and prime factorization is in explaining that a fraction, in base 10, terminates only if the denominator is a power of . Prime factorization comes into play when we realize that this also means the denominator must have only the prime factors and/or . Consider the fraction below:

We multiply by to make the denominator a power of ten in this case, which leads to this fraction terminating. In general if we have we know the fraction terminates and we can use similar logic for the cases where there is a factor of in the denominator.

We run into problems when we try to get a power of in the denominator for a fraction that has a denominator with prime factors other than or . The problem lies in the fact that prime factorization is unique to each number and if we have any prime numbers other than the prime factors of in the denominator we will never be able get the denominator to be a power of . For example we can look at as follows:

We see here that we will never be able to make the in the denominator a power of , thus the decimal representation of this fraction will repeat. Using this knowledge, we are competing with the calculator. We can factor the denominator of any fraction to determine whether or not it will repeat or terminate, more specifically, if the denominator has any prime factors other than or then the fraction repeats. In the classroom going through the explanation of this trick shows students what it means for a fraction to terminate, and how a calculator compares to the students number sense.

**Round 2:** Is a given number a perfect square or not?

Procedure:

* First review what it means to be a perfect square and the properties of exponents.
* Have one of your students get out his/her calculator in order to compete against you.
* Have another student give you a number less than 100, then race the calculator to answer whether or not the given number is a perfect square or not.
* After triumphing over the calculator a few times review the reason that you were able to use prime factorization and number sense to once again defeat the machine.

How it works:

 When we are deciding whether or not a number is a perfect square what we really need to consider is its prime factorization. If we are looking at the prime factorization of a number in order for that number to be a perfect square all of the exponents in the prime factorization must be even. This property becomes clear if we use some properties of exponents and number sense. When we look at a number, let’s call it , if it is a perfect square then we know that it has the form

 where are real numbers.

 However looking at this a little closer, using the properties of exponents, we can see that showing us that in order for a number to be a perfect square its prime factors must all have even powers. This is a good place to emphasize the number sense required to understand why for example , showing that

 When racing against the calculator, the trick is to figure out whether a number happens to be a perfect square or not. Break the number down into its prime factors. In order for the number to be a perfect square we need to see even exponents on all of the prime factors.

 When we go through this explanation there are many different important math concepts that are involved. First we review properties of exponents which are invaluable to a mathematician. We also have shown another place where prime factorization and number sense are used to understand what it means for a number to be a perfect square.

**Round 3:** Can the sum of consecutive integers be a perfect square? If so, what is the smallest one? Can the sum of consecutive numbers be a perfect square?

Procedure:

* Introduce the question from the 2002 12th grade American Mathematics Competition (AMC).
* Discuss what the students see as the best solutions to this problem (most likely using a calculator and using a guess-and-check method).
* Allow time for the students to eventually find that the fourth set of consecutive integers gives you a perfect square.
* Next, challenge the students to try and find a sum of consecutive digits which is a perfect square.
* After a sufficient amount of time show the students why, once again, man triumphs over machine.

How it works:

 This challenge is very useful in emphasizing how important calculators can be and at the same time why calculators in many situations are insufficient for solving problems. When we look at the original problem we can solve it simply by guess and check utilizing the raw computing power of our calculator. We solve the problem by summing the smallest set of consecutive integers and checking to see if that sum is a perfect square, if the sum is a perfect square then we are done, if not, we have to check the second smallest possible sum of consecutive integers. If we elect to utilize this method we will have the following

 This method of guess-and-check utilizing pure calculating power is effective but not very efficient. Now let us try to find the smallest sum of consecutive integers using the same method

 Due to the commutative property of addition when we look at a sum of consecutive integers we can look at it in several different ways; first and most obvious

, where is some positive integer, the sum is equivalent to

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  |  |  |  |  |  |

When we look at this sum we can see that , and so on; there are groups of thus the sum is also equivalent to.

 Here is where prime factorization and number sense help us to prove that there are no consecutive integers whose sum is a perfect square. Since and we know that is always going to be an odd number the prime factorization of our sum will always have a factor of . Since we also know that a perfect square has all prime factors with even exponents, we know that it is not possible for consecutive integers to add up to a perfect square.

 Within the classroom this final activity is a great way to show the clear usefulness and benefit of having the computing power of a calculator in mathematics, but how student problem solving ability and number sense are what drive mathematics. We also show methods for manipulation of sums of integers, show an application of the commutative property of addition and once again provide students with more evidence of the importance of prime numbers and specifically prime factorization.